

On the diameter of the Kronecker product graph*

Fu-Tao Hu, Jun-Ming Xu[†]

School of Mathematical Sciences

University of Science and Technology of China

Wentsun Wu Key Laboratory of CAS

Hefei, Anhui, 230026, China

Abstract: Let G_1 and G_2 be two undirected nontrivial graphs. The Kronecker product of G_1 and G_2 denoted by $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$, two vertices x_1x_2 and y_1y_2 are adjacent if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$. This paper presents a formula for computing the diameter of $G_1 \otimes G_2$ by means of the diameters and primitive exponents of factor graphs.

Keywords: diameter, Kronecker product, primitive exponent

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1 Introduction

For notation and graph-theoretical terminology not defined here we follow [18]. Specifically, let $G = (V, E)$ be a nontrivial graph with no parallel edges, but loops allowed, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set.

For two graphs G and H , Kronecker product $G \otimes H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices x_1x_2 and y_1y_2 are adjacent when $(x_1, y_1) \in E(G)$ and $(x_2, y_2) \in E(H)$.

As an operation of graphs, Kronecker product $G \otimes H$ was introduced first by Weichsel [15] in 1962. It has been shown that the Kronecker product is a good method to construct larger networks that can generate many good properties of the factor graphs (see [9]), and has received much research attention recently. Some properties and graphic parameters have been investigated [1, 2, 5, 8, 11]. The connectivity and diameter are two important parameters to measure reliability and efficiency of a network. Very recently, the connectivity of Kronecker product graph has been deeply studied (see, [3, 6, 7, 11, 12, 14, 16, 17]). However, the diameter of Kronecker product graph has been not investigated yet.

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[†]Corresponding author: xujm@ustc.edu.cn

In this paper, we determine the diameter of Kronecker product graph by means of primitive exponents and diameters of factor graphs. In particular, we obtain that

$$d(G_1 \otimes G_2) = \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\ \max\{\gamma_2 + 1, d_1\} & \text{if } \gamma_1 > \gamma_2; \\ \max\{\gamma_1 + 1, d_2\} & \text{if } \gamma_1 < \gamma_2, \end{cases}$$

where γ_i and d_i are the primitive exponent and diameter of G_i for $i = 1, 2$, respectively.

2 Some Lemmas

Let G be a graph. Denote $\gamma(G; x, y)$ to be the minimum integer such that there exists an (x, y) -walk of length k for any $k \geq \gamma(G; x, y)$ and $\gamma(G)$ be the minimum integer γ for which, for any two vertices x and y in G , there exists an (x, y) -walk of length k for any integer $k \geq \gamma$. Let $\gamma(G) = \max\{\gamma(G; x, y) : x, y \in V(G)\}$.

If $\gamma(G)$ is well-defined, then G is said to be *primitive*, and $\gamma(G)$ is called the *primitive exponent*, *exponent* for short, of G . If $\gamma(G)$ does not exist, then denote $\gamma(G) = \infty$.

Let K_n^+ be a graph obtained from a complete graph K_n by appending a loop on each vertex. It is clear that for a graph G without parallel edges of order n , $\gamma(G) = 1$ if and only if $G \cong K_n^+$.

Let A be the adjacency matrix of G . Equivalently, the exponent of G is the minimum integer γ for which $A^\gamma > 0$ and $A^k \not> 0$ for any positive integer $k < \gamma$. Let A_i be the adjacent matrix of G_i for $i = 1, 2$. Since for any positive integer k , $(A_1 \otimes A_2)^k = A_1^k \otimes A_2^k$, by definition, we have the following result immediately.

Proposition 2.1 *Let G_i be a primitive graph with exponent γ_i for $i = 1, 2$, and $G = G_1 \otimes G_2$. Then $\gamma(G) = \max\{\gamma_1, \gamma_2\}$.*

The following lemmas will be used in proofs of our main results.

Lemma 2.1 (Liu et al. [10]) *A graph G is primitive if and only if G is connected and contains odd cycles.*

Lemma 2.2 (Liu et al. [10]) *Let G be a primitive graph, and let x and y be any pair of vertices in $V(G)$. If there are two (x, y) -walks P_1 and P_2 with lengths k_1 and k_2 , respectively, where k_1 and k_2 have different parity, then $\gamma(G; x, y) \leq \max\{k_1, k_2\} - 1$.*

Lemma 2.3 (Delorme and Solé [4]) *If G is a primitive graph with diameter d , then $\gamma(G) \leq 2d$.*

Lemma 2.4 (Weichesel [15]) *Let G_1 and G_2 be two connected graphs and $G = G_1 \otimes G_2$. Then G is connected if and only if either G_1 or G_2 contains an odd cycle.*

Lemma 2.5 *Let $G = G_1 \otimes G_2$, x_i and y_i be any two vertices in G_i , P_i be an (x_i, y_i) -walk of length ℓ_i in G_i for $i = 1, 2$. If ℓ_1 and ℓ_2 have same parity, then there is an (x_1x_2, y_1y_2) -walk of length $\max\{\ell_1, \ell_2\}$ in G .*

Proof. Without loss of generality, suppose $\ell_1 \geq \ell_2$. Let $k = \ell_1 - \ell_2$. Then k is even. Let $P_1 = (x_1, z_1, \dots, z_{\ell_1-1}, y_1)$ and $P'_2 = (x_2, u_1, \dots, u_{\ell_1-1}, y_2)$ be an (x_2, y_2) -walk of length ℓ_1 in G_2 obtained from P_2 by repeating k times of some edge in P_2 . Then $(x_1x_2, z_1u_1, \dots, z_{\ell_1-1}u_{\ell_1-1}, y_1y_2)$ is an (x_1x_2, y_1y_2) -walk of length ℓ_1 in G . ■

Lemma 2.6 *Let G be a primitive graph with exponent γ and order $n \geq 2$. We have*
(i) if γ is odd, then there exist two vertices x and y , and two different vertices u and v , such that the shortest odd (x, y) -walk and the shortest even (u, v) -walk are of length γ and $\gamma + 1$, respectively;
(ii) if γ is even, then there exist two different vertices p and q , and two vertices w and s , such that the shortest even (p, q) -walk and the shortest odd (w, s) -walk are of length γ and $\gamma + 1$, respectively.

Proof. (i) Assume that γ is odd. If $\gamma = 1$, then $G \cong K_n^+$. Let u and v be two different vertices in G . Then the shortest odd (u, v) -walk and the shortest even (u, v) -walk are of length 1 and 2, respectively. Suppose now $\gamma \geq 3$.

Let A be the adjacency matrix of G . By definition of γ , $A^{\gamma-1} \not\equiv 0$ and $A^{\gamma-2} \not\equiv 0$. These imply that there exist four vertices x, y, u and v such that there are no odd (x, y) -walk and even (u, v) -walk with length $\gamma - 2$ and $\gamma - 1$, respectively. Hence there are no odd (x, y) -walk and even (u, v) -walk with length no more than $\gamma - 2$ and $\gamma - 1$, respectively. Therefore, the shortest odd (x, y) -walk and the shortest even (u, v) -walk are of length γ and $\gamma + 1$, respectively.

We now show $u \neq v$. If $u = v$, then (u, w, u) is an even (u, v) -walk of length 2 for any vertex w adjacent to u in G , a contradiction with $\gamma + 1 \geq 3$.

(ii) Assume γ is even. If $\gamma = 2$, then $d = d(G) = 1$ or 2 since $d \leq \gamma$. If $d = 1$, then G is isomorphic to a complete graph K_n with m vertices having loops and $m < n$. Let p be a vertex with no loop and $q \neq p$ be another vertex in G . Then the shortest even (p, q) -walk and odd (p, p) -walk are of length 2 and 3, respectively. If $d = 2$, then there exist two different vertices p and q such that $d_G(p, q) = 2$, and hence the shortest even (p, q) -walk and odd (p, q) -walk are of length 2 and 3, respectively.

The case when $\gamma > 2$ can be proved by applying the similar discussion as in (i). ■

Lemma 2.7 *Let G_i be a primitive graph with exponent γ_i for $i = 1, 2$, $G = G_1 \otimes G_2$, and $x = x_1x_2$ and $y = y_1y_2$ be two different vertices in G . If the shortest odd (resp. even) (x_1, y_1) -walk in G_1 and the shortest even (resp. odd) (x_2, y_2) -walk in G_2 are of length m and n , respectively, then $d_G(x, y) \geq \min\{m, n\}$.*

Proof. Without loss of generality, assume that m is odd and n is even. Let

$$P = (x_1x_2, \dots, u_1u_2, \dots, y_1y_2)$$

be a minimum (x, y) -path with length s in G . Then

$$(x_1, \dots, u_1, \dots, y_1) \text{ and } (x_2, \dots, u_2, \dots, y_2)$$

be an (x_1, y_1) -walk in G_1 and an (x_2, y_2) -walk in G_2 , respectively, and both of them are of length s .

If s is odd, then $s \geq m$ since the shortest odd (x_1, y_1) -walk in G_1 is of length m ; If s is even, then $s \geq n$ since the shortest even (x_2, y_2) -walk in G_2 is of length n . Therefore $d_G(x, y) = s \geq \min\{m, n\}$. ■

3 Main results

Let G be a connected graph with odd cycles and $\mathcal{C}^o(G)$ be the set of all odd cycles in G . For $C \in \mathcal{C}^o(G)$ and $x \in V(G)$, let

$$d_G(x, C) = \min\{d_G(x, y) : y \in V(C)\},$$

and let

$$\begin{aligned} d_G^o(C) &= \max\{d_G(x, C) : x \in V(G - C)\} \text{ for } C \in \mathcal{C}^o(G), \\ l^o(G) &= \min\{2d_G^o(C) + |V(C)| - 1 : C \in \mathcal{C}^o(G)\}. \end{aligned}$$

We define $l^o(G) = \infty$ if G is bipartite.

Theorem 3.1 $\gamma(G) \leq l^o(G)$ for any connected graph G .

Proof. If G contains no odd cycles, then $l^o(G) = \infty$, and so the conclusion holds. Suppose that G contains odd cycles. By Lemma 2.1, G is primitive. We only need to prove that for any two vertices x and y in G , $\gamma(G; x, y) \leq l^o(G)$.

By definition, there exists an odd cycle C such that $l^o(G) = 2d_G^o(C) + |V(C)| - 1$. Let $d_1 = d_G(x, C)$ and $d_2 = d_G(y, C)$. Then $d_1 \leq d_G^o(C)$ and $d_2 \leq d_G^o(C)$. Let $P_x = (x, x_1, \dots, x_{d_1})$ and $P_y = (y, y_1, \dots, y_{d_2})$ be two shortest paths from x and y to C , respectively, where $x_{d_1}, y_{d_2} \in V(C)$ (maybe $x_{d_1} = y_{d_2}$). Two vertices x_{d_1} and y_{d_2} partition C into two paths P_1 and P_2 with lengths p_1 and p_2 , respectively. Then p_1 and p_2 have different parity, say $p_1 \geq p_2$. Thus, $P_x \cup P_1 \cup P_y$ and $P_x \cup P_2 \cup P_y$ are two (x, y) -walks with length of different parity and at most

$$d_1 + d_2 + p_1 \leq 2d_G^o(C) + |V(C)| = l^o(G) + 1.$$

By Lemma 2.2, $\gamma(G; x, y) \leq l^o(G)$. ■

Corollary 3.1 If G is a connected graph with loops and diameter d , then $\gamma(G) \leq 2d$.

Let $H_{n,p}$ and $F_{n,p}$ ($p \geq 1$) be two graphs, which are obtained by joining a complete graph K_p and a cycle C_p to the end-vertex x_{n-p} of a path $P_{n-p} = (x_1, x_2, \dots, x_{n-p})$ with an edge, respectively.

The following result can be deduced by Theorem 3.1.

Corollary 3.2 (Wang and Wang [13]) Let G be a primitive graph with order n and odd girth $p \geq 3$. Then $\gamma(G) \leq 2n - p - 1$ with equality if and only if G is isomorphic to $F_{n,p}$.

Proof. By Lemma 2.1, G is connected and contains an odd cycle C with $l^o(G) = 2d_G^o(C) + |V(C)| - 1$. Since $d_G^o(C) \leq n - |V(C)|$ and $|V(C)| \geq p$, by Theorem 3.1, we have that

$$\begin{aligned} \gamma(G) &\leq l^o(G) = 2d_G^o(C) + |V(C)| - 1 \\ &\leq 2(n - |V(C)|) + |V(C)| - 1 \\ &\leq 2n - p - 1. \end{aligned} \tag{3.1}$$

The equality implies that all equalities in (3.1) hold, in particular, $d_G^o(C) = n - |V(C)|$ and $|V(C)| = p$. Thus, there is a vertex x_1 such that $d_G(x_1, C) = n - p$ in G . Suppose $P = (x_1, x_2, \dots, x_{n-p}, x_{n-p+1})$ is a shortest path from x_1 to C , where x_{n-p+1} is in C . By the minimality of P and primitivity of G , it is easy to see that G is isomorphic to $F_{n,p}$. Also, if G is isomorphic to $F_{n,p}$, then the shortest odd closed (x_1, x_1) -walk is of length $2(n - p) + p = 2n - p$. This implies there is no closed (x_1, x_1) -walk of length $2n - p - 2$. Hence, $\gamma(F_{n,p}) \geq 2n - p - 1$. ■

Corollary 3.3 *If $p \geq 3$, then $\gamma(H_{n,p}) = 2n - 2p + 2$.*

Proof. Let $G = H_{n,p}$. Since G contains K_p and $p \geq 3$, G is primitive by Lemma 2.1, and so $d_G^o(C) = d_G(x_1, C) = n - p$ for any $C \in \mathcal{C}^o(G)$. Let C be a cycle of length 3 in G . By Theorem 3.1,

$$\gamma(G) \leq l^o(G) \leq 2d_G^o(C) + |V(C)| - 1 = 2(n - p) + |V(C)| - 1 \leq 2n - 2p + 2.$$

It is clear that the shortest odd closed (x_1, x_1) -walk is of length $2(n - p) + 3$. This implies there is no closed (x_1, x_1) -walk of length $2(n - p) + 1$. Hence, $\gamma(G) \geq 2n - 2p + 2$. The conclusion follows. \blacksquare

Theorem 3.2 *Let G_i be a connected graph with diameter $d_i \geq 1$ and exponent $\gamma_i = \gamma(G_i)$ for $i = 1, 2$, G_1 contains odd cycles, and $G = G_1 \otimes G_2$. Then the diameter $d(G)$ of G satisfies the following properties.*

- (1) $d(G) \geq \max\{d_1, d_2\}$.
- (2) If G_2 contains odd cycles, then

$$d(G) \geq \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\ \min\{\gamma_1, \gamma_2\} + 1 & \text{if } \gamma_1 \neq \gamma_2. \end{cases}$$

- (3) $d(G) \leq \max\{\gamma_1, \gamma_2\}$.

- (4) $d(G) \leq \min\{\max\{\gamma_1 + 1, d_2\}, \max\{\gamma_2 + 1, d_1\}\}$ with equality if G_2 is bipartite.

Proof. Since both G_1 and G_2 are connected and G_1 contains odd cycles, by Lemma 2.1 and Lemma 2.5, γ_1 is well-defined and G is connected. Since $d_1 \geq 1$ and $d_2 \geq 1$, the order of G_1 and G_2 are no less than 2.

(1) For $i = 1, 2$, let x_i and y_i be two vertices in G_i with $d_{G_i}(x_i, y_i) = d_i$ and let $P = (x_1x_2, \dots, u_1u_2, \dots, y_1y_2)$ be a shortest (x_1x_2, y_1y_2) -path in G . Then $(x_1, \dots, u_1, \dots, y_1)$ and $(x_2, \dots, u_2, \dots, y_2)$ are two walks in G_1 and G_2 , respectively. Thus $d(G) \geq d(P) \geq \max\{d_1, d_2\}$.

(2) Since G_2 contains odd cycles, γ_2 is well-defined by Lemma 2.1. Without loss of generality, assume $\gamma_2 \geq \gamma_1$ and γ_1 is odd. By Lemma 2.6, there exist two different vertices x_1 and y_1 such that the shortest even (x_1, y_1) -walk is of length $\gamma_1 + 1$ in G_1 ; also there exist two vertices x_2 and y_2 such that the shortest odd (x_2, y_2) -walk is of length γ_2 or $\gamma_2 + 1$ in G_2 . By Lemma 2.7, $d_G(x_1x_2, y_1y_2) \geq \min\{\gamma_1 + 1, \gamma_2\}$, and so

$$d(G) \geq \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\ \min\{\gamma_1, \gamma_2\} + 1 & \text{if } \gamma_1 \neq \gamma_2. \end{cases}$$

(3) Without loss of generality, suppose that γ_2 is well-defined and $\gamma_2 \leq \gamma_1$. Let $x = x_1x_2$ and $y = y_1y_2$ be any two different vertices in G . By definition of γ , there exist an (x_1, y_1) -walk and an (x_2, y_2) -walk of length γ_1 in G_1 and G_2 , respectively. By Lemma 2.5, there exists an (x, y) -walk of length γ_1 , and hence $d(G; x, y) \leq \gamma_1$. By the arbitrariness of x and y , we have $d(G) \leq \gamma_1$.

(4) Without loss of generality, suppose that γ_2 is well-defined, and only need to prove $d(G) \leq \max\{\gamma_1 + 1, d_2\}$. Let $x = x_1x_2$ and $y = y_1y_2$ be any two different vertices in G and $d'_2 = d_{G_2}(x_2, y_2)$ (maybe $x_2 = y_2$). If $d'_2 \geq \gamma_1$, then there exists an (x_1, y_1) -walk of length d'_2 in G_1 by definition of γ . By Lemma 2.5, there exists an (x, y) -walk of length

d'_2 in G . If $d'_2 < \gamma_1$, then one of $d'_2 + \gamma_1$ and $d'_2 + \gamma_1 + 1$ is even. By definition of γ , there exist two (x_1, y_1) -walks of lengths γ_1 and $\gamma_1 + 1$ in G_1 , respectively. By Lemma 2.5, there exists an (x, y) -walk of length no more than $\gamma_1 + 1$ in G . Thus $d_G(x, y) \leq \max\{\gamma_1 + 1, d_2\}$, and hence $d(G) \leq \max\{\gamma_1 + 1, d_2\}$ by arbitrariness of x and y .

Now assume that G_2 is bipartite. Let x_2 and y_2 be two vertices in different parts in G_2 . Then any (x_2, y_2) -walk and closed (x_2, x_2) -walk are of odd and even length in G_2 , respectively. If $\gamma_1 = 1$, then $d(G) \geq d_G(x_1x_2, y_1x_2) \geq 2 = \gamma_1 + 1$ for any two different vertices x_1 and y_1 in G_1 since $|V(G_1)| \geq 2$. Next, assume $\gamma_1 \geq 2$.

By using the Lemma 2.6, we have the following conclusions. If γ_1 is odd, then there exist two different vertices x_1 and y_1 such that the shortest even (x_1, y_1) -walk is of length $\gamma_1 + 1$ in G_1 , and hence $d(G) \geq d_G(x_1x_2, y_1x_2) \geq \gamma_1 + 1$. If γ_1 is even, then there exist two vertices x_1 and y_1 such that the shortest odd (x_1, y_1) -walk is of length $\gamma_1 + 1$ in G , and hence $d(G) \geq d_G(x_1x_2, y_1y_2) \geq \gamma_1 + 1$. By the conclusion (1), $d(G) \geq d_2$, and hence $d(G) = \max\{\gamma_1 + 1, d_2\}$.

The theorem follows. \blacksquare

Corollary 3.4 *Let G_i be a connected graph with diameter $d_i \geq 1$ and $l_i = l^o(G_i)$ for $i = 1, 2$, $G = G_1 \otimes G_2$. Then*

$$d(G) \leq \min\{\max\{l_1 + 1, d_2\}, \max\{l_2 + 1, d_1\}\}.$$

Proof. Without loss of generality, we can suppose that both G_1 and G_2 contain odd cycles. By Theorem 3.1, $\gamma(G_1) \leq l_1$ and $\gamma(G_2) \leq l_2$. The conclusion follows by the conclusion (4) in Theorem 3.2. \blacksquare

Corollary 3.5 *Let G_i be a connected graph with diameter $d_i \geq 1$ for $i = 1, 2$ and $G = G_1 \otimes G_2$. If G_1 contains odd cycles, then $d(G) \leq \max\{2d_1 + 1, d_2\}$.*

Proof. By Lemma 2.3, $\gamma(G_1) \leq 2d_1$. The Theorem follows by the conclusion (4) in Theorem 3.2. \blacksquare

The following result, obtained by Leskovec et al. [9], can be deduced by Theorem 3.2 immediately.

Corollary 3.6 (Leskovec et al. [9]) *Let G_i be a connected graph with diameter $d_i \geq 1$ and there is a loop on every vertex of G_i for $i = 1, 2$. Then $d(G_1 \otimes G_2) = \max\{d_1, d_2\}$.*

Proof. It is clear that $\gamma(G_1) = d_1$ and $\gamma(G_2) = d_2$ since each of G_1 and G_2 has a loop on every vertex. The conclusion follows by the conclusions (1) and (3) in Theorem 3.2. \blacksquare

Corollary 3.7 *Let G be a primitive graph with order $n \geq 2$. Then $\gamma(G) = d(G \otimes K_2) - 1$.*

By Theorem 3.2, we immediately obtain our main results in this paper.

Theorem 3.3 *Let G_i be a connected graph with diameter $d_i \geq 1$ and exponent $\gamma_i = \gamma(G_i)$ for $i = 1, 2$. If G_1 contains odd cycles, then*

$$d(G_1 \otimes G_2) = \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\ \max\{\gamma_2 + 1, d_1\} & \text{if } \gamma_1 > \gamma_2; \\ \max\{\gamma_1 + 1, d_2\} & \text{if } \gamma_1 < \gamma_2. \end{cases}$$

In Theorem 3.3, we consider the diameter of the Kronecker product of two graphs G_1 and G_2 with order no less than 2. Next, we consider the case that at least one of G_1 and G_2 with order 1. Let G be a connected graph with order n and no parallel edges. We have noted in Section 2, $\gamma(G) = 1$ if and only if $G \cong K_n^+$. For a graph H with order 1, if $G \otimes H$ is connected, then $H \cong K_1^+$ since $G \otimes K_1$ is empty. It is easy to see that $K_1^+ \otimes G \cong G$ and then $d(K_1^+ \otimes G) = d(G)$.

In the following, we show the diameters for some special Kronecker product of two graphs only by using the diameters of factor graphs.

Theorem 3.4 *Let G_i be a connected graph with order $n_i \geq 2$ for $i = 1, 2$. Then $d(G_1 \otimes G_2) = 1$ if and only if $G_1 \cong K_{n_1}^+$ and $G_2 \cong K_{n_2}^+$.*

Proof. The sufficiency is obviously.

Now we show the necessity. By contradiction. Without loss of generality, assume $G_1 \not\cong K_{n_1}^+$. Then either there exists a vertex x such that it does not contain a loop or $d(G_1) \geq 2$. Then $d(G) \geq d_G(xy, xz) \geq 2$ for any two different vertices $y, z \in V(G_2)$ or $d(G) \geq d(G_1) \geq 2$ by the conclusion (1) in Theorem 3.2. ■

Theorem 3.5 *Let $G \not\cong K_n^+$ be a connected graph with order $n \geq 2$ and $m \geq 2$. Then*

$$d(K_m^+ \otimes G) = \begin{cases} 2, & d(G) = 1; \\ d(G), & d(G) \geq 2. \end{cases}$$

Proof. The Theorem follows by Theorem 3.3 since $\gamma(K_m^+) = 1$ and $\gamma(G) \geq 2$. ■

Theorem 3.6 *Let G be a connected graph with diameter $d \geq 1$ and H be a complete t partite graph with $t \geq 3$. Then*

$$d(G \otimes H) = \begin{cases} d, & d \geq 3; \\ 2, & d \leq 2 \text{ and } \gamma(G) \leq 2; \\ 3, & d \leq 2 \text{ and } \gamma(G) > 2. \end{cases}$$

Proof. It is clear that $d(H) \geq 2$, H is primitive and $\gamma(H) = 2$. The Theorem follows by Theorem 3.3. ■

Corollary 3.8 *Let G be $H_{n,p}$ or $F_{n,p}$ with odd cycles and diameter $d_1 \geq 1$, and H be any connected graph with diameter $d_2 \geq 1$.*

- (1) *If H is bipartite, then $d(G \otimes H) = \max\{2d_1 + 1, d_2\}$.*
- (2) *If $H = G_{n_2, p_2}$ is non-bipartite, then*

$$d(G \otimes H) = \begin{cases} 2d_1 & \text{if } d_1 = d_2; \\ \max\{d_1, 2d_2 + 1\} & \text{if } d_1 > d_2; \\ \max\{d_2, 2d_1 + 1\} & \text{if } d_1 < d_2. \end{cases}$$

Proof. By Lemma 2.1, G is primitive since G contains odd cycles. By Lemma 2.5, $G \otimes H$ is connected. By Corollary 3.2 and 3.3, $\gamma(G) = 2d_1$. If H is bipartite, then H is not primitive by Lemma 2.1. Thus $\infty = \gamma(H) > \gamma(G)$, and hence, $d(G \otimes H) = \max\{2d_1 + 1, d_2\}$ by Theorem 3.3. If $H = G_{n_2, p_2}$ is non-bipartite, then $\gamma(H) = 2d_2$. The conclusion follows by Theorem 3.3 immediately. ■

Corollary 3.9 *Let C_m be an odd cycle and H be a connected graph with order n and diameter $d \geq 1$.*

(1) If H is bipartite, then $d(C_m \otimes H) = \max\{m, d\}$. Hence $d(C_m \otimes P_n) = \max\{m, n-1\}$, and $d(C_m \otimes C_n) = \max\{m, \frac{n}{2}\}$ if n is even.

(2) If $H = C_n$ and n is odd, then

$$d(C_m \otimes C_n) = \begin{cases} m-1 & \text{if } m = n, \\ \max\{n, \frac{m-1}{2}\} & \text{if } m > n \\ \max\{m, \frac{n-1}{2}\} & \text{if } m < n. \end{cases}$$

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